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THE 'NO BOUNDARY CONDITION' OUTFLOW BOUNDARY CONDITION

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SUMMARY

We use a one-dimensional model problem of advection–diffusion to investigate the treatment recently advocated by Papanastasiou and colleagues to deal with boundary conditions at artificial outflow boundaries.

Using finite elements of degree *p*, we show that their treatment is equivalent to imposing the condition that the $(p+1)$ st derivative of the dependent variable should vanish at a point close to the outflow. This is then shown to + 1)st derivative of the dependent variable should vanish at a point close to the outflow. This is then shown to d to errors of order $\mathcal{O}((h+1/Pe)^{p+1})$ in the numerical solutions (where *h* is the maximum element size lead to errors of order $\mathcal{O}((h P \rho \text{ is the global Peclet nu})$ $+1/Pe^{p+1}$) in the numerical solutions (where *h* is the maximum element size and
mber), which is superior to the errors of order $O(h^{p+1} + 1/Pe)$ obtained using and
ition. These findings are verified by numerical experim *Pe* is the global Peclet number), which is superior to the errors of order $\mathcal{O}(h^{p+1} + 1/Pe)$ obtained using a standard no-flux outflow condition. These findings are verified by numerical experiments.
KEY WORDS: advect standard no-flux outflow condition. These findings are verified by numerical experiments.

KEY WORDS: advection–diffusion; outflow boundary conditions; finite elements

1. A MODEL PROBLEM IN 1D

We consider the advection–diffusion problem

$$
T_t + uT_x = \varepsilon T_{xx} + S \tag{1}
$$

 $uT_x = \varepsilon T_{xx} + S$ (1)
with given initial data $T(x, 0) = T_0(x), 0 < x < \infty$, and a
i, without loss of generality we may take $T(0, t) = 0, t > 0$.
domain and that ε is a positive constant.
ry to restrict the length of the inter on the quarter-plane Dirichlet boundary condition (BC) at *x* We assume that $u(x, t) > 0$ throughout the domain and that ε is a positive constant.

{0 < *x* < ∞, *t* > 0}, with given initial data *T*(*x*, 0) = *T*₀(*x*), 0 < *x* < ∞, and a nodition (BC) at *x* = 0; without loss of generality we may take *T*(0, *t*) = 0, *t* > 0. *t*) > 0 throughout the domain and t = 0; without loss of generality we may take $T(0, t) = 0, t > 0$.
t the domain and that ε is a positive constant.
essary to restrict the length of the interval to some finite value
n has then to be imposed at $x = L$ (in or $(x, t) > 0$ throughout the domain and that ε is a positive constant.

ral purposes it is necessary to restrict the length of the interval to s

al boundary condition has then to be imposed at $x = L$ (in order to

has a u For computational purposes it is necessary to restrict the length of the interval to some finite value *L*, say. An artificial boundary condition has then to be imposed at $x = L$ (in order to ensure that the $=L$ (in order to ensure that the
ficantly affect the solution in the
initial data $T_0(x)$ are such that
ently large that the solution is
x' condition truncated problem has a unique solution) in such a way as to not significantly affect the solution in the interior. If it is assumed that the source term $S(x, t)$ and the initial data $T_0(x)$ are such that *x*; *t*) and the initial data *T*₀*x*) are such that that *L* is sufficiently large that the solution is appose the 'no-flux' condition $x = L$. (2) epends not only on *L* being sufficiently large but *T*constant in space for $L \leq x < \infty$, then one may impose the 'no-flux' condition

$$
T_x = 0 \quad \text{at } x = L. \tag{2}
$$

 $(x, t) \rightarrow$ constant as $x \rightarrow \infty$ for each $t > 0$ and that *L* is sufficiently large that the solution is
onstant in space for $L \le x < \infty$, then one may impose the 'no-flux' condition
 $T_x = 0$ at $x = L$. (2)
his has been the trad ∞ , then one may impose the 'no-flux' condition
 $T_x = 0$ at $x = L$.

proach. Its success depends not only on L being s

ve so that any signal is damped as it is convection- $= 0$ at $x = L$. (2)
access depends not only on *L* being sufficiently large but
ny signal is damped as it is convected in the positive
an outflow boundary condition on a multidimensional
not that a condition of the form This has been the traditional approach. Its success depends not only on *L* being sufficiently large but also on ε being strictly positive so that any signal is damped as it is convected in the positive x direction *x*-direction.

Lohéac¹ analyses the effect of imposing an outflow boundary condition on a multidimensional

$$
T_t + uT_x = 0 \quad \text{at } x = L \tag{3}
$$

form of equation (1) with *S* = 0. It is proved that a condition of the form
 $T_t + uT_x = 0$ at $x = L$

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 $+ uT_x = 0$ at $x = L$ (3)
 Received 12 September 1995
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leads to an error (difference between the problems on the semi-infinite real line and the finite interval $[0, L]$) of order $\mathcal{O}(\varepsilon^2)$ provided that $|u| > 0$. (Our Lemma 1 with $p = 1$ proves a similar result in the steady case.)

The use of extrapolation conditions of the form

$$
\frac{\partial^j T}{\partial x^j} = 0 \quad \text{at } x = L \tag{4}
$$

for the Navier–Stokes equations is described by Johansson² and Nordström,³ the former recommending the use of $i = 3$ while the latter analyses the case $i = 2$.

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ent methods (FEMs). It = 3 while the latter analyses the case $j = 2$.
erent approach that has recently been prop
or finite element methods (FEMs). It differs
eek to impose an outflow BC on the cont
te approximation of it.
ch, we require the wea We shall analyse a different approach that has recently been proposed by Papanastasiou and colleagues $4-6$ specifically for finite element methods (FEMs). It differs from the methods described above in that it does not seek to impose an outflow BC on the continuous problem but does so implicitly within the discrete approximation of it.

To describe their approach, we require the weak form of equation (1) on the interval $0 < x < L$. This is such that $T(x, t)$ must satisfy the infinite system of ordinary differential equations (ODEs)

$$
(\phi, T_t) + (\phi, uT_x) + \varepsilon(\phi_x, T_x) = (\phi, S) + \varepsilon \phi T_x|_{x=L}
$$
\n
$$
(5)
$$

(*x*, *t*) must satisfy the infinite system of ordinary differential equations (ODEs)
 $(\phi, T_t) + (\phi, uT_x) + \varepsilon(\phi_x, T_x) = (\phi, S) + \varepsilon \phi T_x|_{x=L}$ (*z*)
 $\cap {\phi(0) = 0}$, together with the given initial data and the boundary condition (ϕ , T_t) + (ϕ , uT_x) + $\varepsilon(\phi_x, T_x) = (\phi, S) + \varepsilon \phi T_x|_{x=L}$ (5)
(0) = 0}, together with the given initial data and the boundary condition
2), then the boundary term on the right of (5) vanishes, so that
(ϕ , T_t) + for all $\phi \in H^1(0, L) \cap {\phi(0) = 0}$, together with the given initial data and the boundary condition $T = 0$ at $x = 0$.
If we impose the BC (2), then the boundary term on the right of (5) vanishes, so that
 $(\phi, T_t) + (\phi, uT_x) + \v$ $T = 0$ at $x = 0$.

If we impose the BC (2), then the boundary term on the right of (5) vanishes, so that

$$
(\phi, T_t) + (\phi, uT_x) + \varepsilon(\phi_x, T_x) = (\phi, S),
$$
\n(6)

for which (2) is a natural BC.

 $= 0$ at $x = 0$.
If we impose $(\phi, T_t) + (\phi, uT_x) + \varepsilon(\phi_x, T_x) = (\phi, S),$ (6)
stasiou *et al.* suggest that the weak form (5) should be left as it is and
be treated as being unknown—this is termed in Reference 5 a *free*
eling that it should be referred to as In contrast with this, Papanastasiou *et al.* suggest that the weak form (5) should be left as it is and that the boundary term should be treated as being unknown—this is termed in Reference 5 a *free boundary condition*. It is our feeling that it should be referred to as the '*no BC' boundary condition* as this more accurately describes the situation, since within a purely continuous setting (as opposed to finite element approximations) the weak formulation (5) is invalid because it is equivalent to not setting any BC at $x = L$ and the governing equations (5) cannot therefore isolate a unique solution. The aim of this paper is to describe the behaviour of finite element approximations of (5) and, in particular, to ascertain what numerical BCs are implied by this weak form.

setting any BC at $x = L$ and the governing equations (5) cannot therefore isolate a unique solution.
The aim of this paper is to describe the behaviour of finite element approximations of (5) and, in particular, to ascerta In the next section we describe the FE methods and this is followed in Section 3 by the derivation of a boundary condition which is implied by the FE equations. It is shown that finite elements of $=p+1$, except that these hold at some point
 $= 1$ (linear elements) is also equivalent to (3).
 py condition leads to errors of order $\mathcal{O}(e^{p+1})$ at

is contrasts with the errors of order $\mathcal{O}(e)$ when
 y a sele within the last element rather than at *x* = *L*. The case $p = 1$ (linear elements) is also equivalent to (3).
the new boundary condition leads to errors of order $\mathcal{O}(e^{p+1})$ at
of degree *p*. This contrasts with the errors of order $\mathcal{O}(e)$ when
lts are veri We also show that for steady problems the new boundary condition leads to errors of order $\mathcal{O}(e^{p+1})$ at the outflow when using finite elements of degree p. This contrasts with the errors of order $\mathcal{O}(e)$ when using the outflow when using finite elements of degree *p*. This contrasts with the errors of order $\mathcal{O}(\varepsilon)$ when using the no-flux condition. These results are verified by a selection of steady and unsteady numerical exam using the no-flux condition. These results are verified by a selection of steady and unsteady numerical examples in Section 4.

In Section 5 we avoid the unorthodox implied boundary condition derived in Section 3 by establishing the equivalence of the 'no BC' FE equations and the Galerkin FE approximation to a problem with standard Dirichlet–Neumann-type BCs.

2. FE APPROXIMATION

Because there is substantial interest in the *p*-version as well as the *h*-version of the FEM, we shall adopt an FE approximation using continuous piecewise polynomials of any degree $p \ge 1$. Our conclusions will be independent of the exact nature of the basis functions employed, but for the sake of simplicity we assume it to be the classic nodal (or Lagrangian) basis. With the imposition of the BC *T*(0, *t*) = 0 the dimension of the resulting space is *pN*. Thus, if we divide the interval $0 < x < L$
into *N* elements by the knots
 $0 = x_0 < x_1 < \cdots < x_N = L$,
then each element has $p - 1$ nodes in its interior and one node into *N* elements by the knots

$$
0 = x_0 < x_1 < \cdots < x_N = L
$$

 $x_0 < x_1 < \cdots < x_N = L$,
 its interior and one node at
 x we define
 $\max_{x \in N} (x_j - x_{j-1}), \quad h_N = x_N$ then each element has $p-1$ nodes in its interior and one node at each of its endpoints. The knots are not required to be equally spaced and we define not required to be equally spaced and we define

$$
h = \max_{i \le j \le N} (x_j - x_{j-1}), \quad h_N = x_N - x_{N-1}.
$$

 $=\max_{i\leq j\leq l}$
a this gr $(x_j - x_{j-1}),$ $h_N = x_N - x_{N-1}.$
 id by $\{\phi_1, \phi_2, ..., \phi_{pN}\},$ then t
 $h(x, t) = \sum_{r=1}^{pN} T_i(t)\phi_i(x),$ Denoting the basis functions on this grid by $\{\phi_1, \phi_2, ..., \phi_{pN}\}\)$, then the FE approximation *Th* to *T* takes the form
 $T^h(x, t) = \sum_{j=1}^{pN} T_j(t)\phi_j(x)$, (7)

where *T*_{*i*} denotes the value of *Th* at the *i*th node. Th takes the form

$$
T^{h}(x, t) = \sum_{j=1}^{pN} T_{j}(t)\phi_{j}(x),
$$
\n(7)
\ne *j*th node. The FE approximation of (6) then leads to the set of
\n
$$
0 + \varepsilon(\phi_{j_{x}}, T^{h}_{x}) = (\phi_{j}, S), \quad j = 1, 2, ..., pN.
$$
\n(8)

 $=$ 1 Γ h where T_j denotes the value of T^h at the *j*th node. The FE approximation of (6) then leads to the set of *pN* ODEs

$$
(\phi_j, T_t^h) + (\phi_j, uT_x^h) + \varepsilon(\phi_{j_x}, T_x^h) = (\phi_j, S), \quad j = 1, 2, ..., pN.
$$
 (8)

On the other hand, the weak form (5) leads to

$$
(\phi_j, T_t^n) + (\phi_j, uT_x^n) + \varepsilon(\phi_{j_x}, T_x^n) = (\phi_j, S), \quad j = 1, 2, ..., pN.
$$
\nOn the other hand, the weak form (5) leads to

\n
$$
(\phi_j, T_t^n) + (\phi_j, uT_x^n) + \varepsilon(\phi_{j_x}, T_x^n) = (\phi_j, S) + \varepsilon\phi_j T_x^n|_{x=L}, \quad j = 1, 2, ..., pN.
$$
\n(9)

\nThe discrete equations (8) and (9) differ only in the boundary term and consequently the ODEs they

 (φ_j)
equenter of φ_j
explients $f(\phi_j, uT_x^n) + \varepsilon(\phi_{j_x}, T_x^n) = (\phi_j, S) + \varepsilon\phi_j T_x^n|_{x=L}, \quad j = 1, 2, ..., pN.$ (9)

ons (8) and (9) differ only in the boundary term and consequently the ODEs they

for the index $j = pN$ associated with nodes lying on the artificial bou generate differ only for the index $j = pN$ associated with nodes lying on the artificial boundary $x = L$. *pN* associated with nodes lying on the artificial boundary *x* = *L*.

nodes in the last element are $x_{l+1}, x_{l+2},...,x_{l+p}$ and the
 $\phi_{l+1}, \phi_{l+2},..., \phi_{l+p}$ are polynomials (rather than piecewise

ttly, for $j = l + 1, l + 2, ..., l$ Defining $l = p(N - 1)$, then the nodes in the last element are $x_{l+1}, x_{l+2}, \ldots, x_{l+p}$ and the *p*(*N* - 1), then the nodes in the last element are $x_{l+1}, x_{l+2},..., x_{l+p}$ and the ng basis functions $\phi_{l+1}, \phi_{l+2},..., \phi_{l+p}$ are polynomials (rather than piecewise s) on (x_l, L) . Consequently, for $j = l + 1, l + 2, ..., l + p$, (9)

corresponding basis functions
$$
\phi_{l+1}, \phi_{l+2}, \dots, \phi_{l+p}
$$
 are polynomials (rather than piecewise
polynomials) on (x_l, L) . Consequently, for $j = l + 1, l + 2, \dots, l + p$, (9) is *exactly* equivalent to

$$
\int_{x_l}^{L} \phi_j(T_l^h + uT_x^h - \varepsilon T_{xx}^h - S)dx = 0, \quad j = l + 1, l + 2, \dots, l + p,
$$
 (10)
which holds for all degrees $p \ge 1$. Note that (10) does not hold for $j < l + 1$.
3. AN IMPLIED BOUNDARY CONDITION

which holds for all degrees $p \ge 1$. Note that (10) does not hold for $j < l$

3. AN IMPLIED BOUNDARY CONDITION

 $+1.$ Before turning to finite elements of order *p*, we look first at the simpler situations of linear (*p* = 1) and quadratic (*p* = 2) elements.
Linear elements
When $p = 1$, we have $T_{xx}^h \equiv 0$ for $x \in (x_l, L)$ and so equati quadratic $(p=2)$ elements.

Linear elements

= 2) elements.
ents
: 1, we have T_x^l *L*

$$
\equiv 0 \text{ for } x \in (x_l, L) \text{ and so equations (9) give}
$$
\n
$$
\int_{x_l}^{L} \phi_N(T_l^h + u T_x^h - S) dx = 0.
$$
\n
$$
0 \text{ for } x \in (x_l, L), \text{ we may apply the mean val}
$$
\n
$$
T_l^h + u T_x^h = S
$$

When $p = 1$, we have T_{xx}^h
 xx thermore, since $\phi_N(x) >$

duce that Furthermore, since $\phi_N(x) > 0$ for $x \in (x_l, L)$, we may apply the mean value theorem for integrals to deduce that
 $T_t^h + uT_x^h = S$ (11) deduce that

$$
T_t^h + uT_x^h = S \tag{11}
$$

at some point $\xi \in (x_i, L)$. Clearly $\xi \to L$ as $h \to 0$, so that this may be taken as a form of BC in the neighbourhood of $x = L$. In general ξ varies with t.

 $\in (x_1, L)$. Clearly $\xi \to L$ as $h \to 0$, so that this may be taken as a form of BC in the
of $x = L$. In general ξ varies with t .
both linear polynomials in x on the interval (x_l, L) , the point ξ may be identified *L*. In general ξ varies with *t*.
th linear polynomials in *x* on the $\equiv T_t^h + uT_x^h - S$ is then a linear
on (x_l, L) . Consequently,
 $R = a(t)(x -$ If *S* and *u* are both linear polynomials in *x* on the interval (x_i, L) , the point ξ may be identified, (x_l, L) , the point ξ may be identified,
al in *x* (for each *t*) which is orthogonal
BC' boundary condition implies the to $\phi_N = (x - x_l)/h_N$ on (x_l, L) . Consequently,

$$
R = a(t)(x - L + hN/3)
$$

since the residual $R \equiv T_t^h + uT_x^h$
to $\phi_N = (x - x_l)/h_N$ on (x_l, L) .
for some function $a(t)$. Thus
satisfaction of the 'reduced equa *S* is then a linear polynomial in *x* (for each *t*) which is orthogonal
 R = $a(t)(x - L + h_N/3)$
 $\xi = L - h_N/3$ and the 'no BC' boundary condition implies the

tion' (11)₇ at the fixed location $x = L - h_N/3$. A similar observ $R = (x - x_l)/h_N$ on (x_l, L) . Consequently,
 $R = a(t$
 me function *a*(*t*). Thus $\xi = L - h_N/3$

ction of the 'reduced equation' (11) at t

en made by Heinrich and Vionnet,⁷ alt $a(t)(x - L + h_N/3)$
h_N/3 and the 'no 1
(1) at the fixed location
 \int_0^{π} , although they do n for some function *a*(*t*). Thus $\xi = L - h_N/3$ and the 'no BC' boundary condition implies the duced equation' (11) at the fixed location $x = L - h_N/3$. A similar observation inrich and Vionnet,⁷ although they do not identify the point ξ .
We satisfaction of the 'reduced equation' (11) at the fixed location *x* has been made by Heinrich and Vionnet,⁷ although they do not identify the point ξ .

Quadratic elements

two discrete equations pertaining to the element of degree one and two in *x* respectively on (x_i, L) whose coefficients may depend on *t*.

Let is a similar observation the 'reduced equation' (11) at the fixed location $x = L - h_N/3$. A similar observation

s been made by Heinrich and Vionnet,⁷ although they do not identify the point ξ .
 Let indratic elem ≡ constant for *x* ∈ (*x_i*, *L*) and (10) holds for *j* = *l* + 1, *l* + 2—there are
ning to the element (*x_i*, *L*). We shall assume that *u* and *S* are polynomials
respectively on (*x_i*, *L*) whose coefficients (x_l, L) . We shall assume that *u* and *S* are polynomials *L*) whose coefficients may depend on *t*.

that is orthogonal on (x_l, L) to both ϕ_{l+1} and ϕ_{l+2} , it of degree two that vanish at the left endpoint $x = x_l$ (x_l, L) whose coefficients may depend on *t*.
 x that is orthogonal on (x_l, L) to both ϕ_{l+1} ials of degree two that vanish at the left end
 $X - 1$, $X = -1 + 2\frac{x - x_l}{h_N}$, Since *R* is a polynomial of degree two in *x* that is orthogonal on (x_i, L) to both ϕ_{i+1} and ϕ_{i+2} , it (x_l, L) to both ϕ_{l+1} and ϕ_{l+2} , it
anish at the left endpoint $x = x_l$
 $\frac{-x_l}{h_N}$, must therefore be orthogonal to all polynomials of degree two that vanish at the left endpoint $x = x_l$
of the outflow element. It follows that
 $R = a(t)(5X^2 - 2X - 1)$, $X = -1 + 2\frac{x - x_l}{h_N}$,
where *a* is some function of *t* alo of the outflow element. It follows that

$$
R = a(t)(5X^2 - 2X - 1), \quad X = -1 + 2\frac{x - x_l}{h_N},
$$

of *t* alone (different from that in the linear case).
 $\xi_1, \xi_2 \in (x_l, L)$, where
 $\xi_{1,2} = x_l + h_N(1/5 \pm \sqrt{6/10}).$

where *a* is some function of *t* alone (different from that in the linear case). Thus the residual must be zero at the two points $x = \xi_1, \xi_2 \in (x_l, L)$, where

$$
\xi_{1,2} = x_l + h_N(1/5 \pm \sqrt{6/10}).
$$

= ξ_1 , $\xi_2 \in (x_l, L)$, where
 $\xi_{1,2} = x_l + h_N$

r case, these conditions of

ation (1) is exactly satis

ver, using the facts that ξ $x_l + h_N(1/5 \pm \sqrt{6/10}).$

onditions do not provide B

actly satisfied at these pc

facts that $\partial R/\partial x = 0$ at X = In contrast with the linear case, these conditions do not provide BCs for the system but merely show that the differential equation (1) is exactly satisfied at these points—the residual is collocated at *x* = ξ_1 and *x* = ξ_2 . However, using the facts that $\partial R/\partial x = 0$ at *X* = 1/5 and T_{xx}^h = constant on *(x_l*, *L*), it follows that $\frac{\partial}{\partial x}(T_t^h + uT_x^h - S)\Big|_{x=\xi} = 0$ (12)
at $x = \xi = x_l + 3h_N/5$. This condition i it follows that $\frac{1}{1}$

$$
\frac{\partial}{\partial x}(T_t^h + u T_x^h - S)\Big|_{x=\xi} = 0\tag{12}
$$

.
th
m at *x* as being the BC implied by (9).

 $(T_t^h$
s in
ly v
wh $+ uT_x^h - S$ $\Big|_{x=\xi} = 0$ (12)

dependent of the PDE (1) and may therefore be construed

veaker result may be given for the cases where *S* is not

veaker result may be given for the cases where *S* is not

veaker is not l $\zeta = \zeta = x_l + 3h_N/5$. This condition is independent of the PDE (1) and may therefore be construed
eing the BC implied by (9).
In alternative derivation of a slightly weaker result may be given for the cases where S is not
 An alternative derivation of a slightly weaker result may be given for the cases where *S* is not necessarily a quadratic polynomial or where u is not linear in x . We begin by scaling the basic functions to have 'unit mass':

$$
\hat{\phi}_j(x) = \phi_j(x) / \int_0^L \phi_j(t) dt, \quad j = l + 1, l + 2;
$$
\n(13)

 $\frac{\varphi}{n}$
on: *p*_j(*x*) = *d*_j(*x*) | $\int_0^b \phi_j(t)dt$, $j = l + 1, l + 2$; (13)
then clearly the FE equations (8)–(10) continue to hold with ϕ_j replaced by $\hat{\phi}_j$. Subtracting the two
resulting equations corresponding to $j = l + 1, l +$ (x)
 (x) c
 $=$ then clearly the FE equations (8)–(10) continue to hold with ϕ_i replaced by $\hat{\phi}_i$. Subtracting the two resulting equations corresponding to $j = l + 1$, $l + 2$ from each other gives (using T_{xx}^h = constant on $(\hat{\phi}_{l+1} - \hat{\phi}_{l+2}, T_l^h + uT_x^h - S) = 0$. (14)
Defining $\psi(x) = \int_{x_l}^x (\hat{\phi}_{l+1}(t) - \hat{\phi}_{l+2}(t))dt$, then it is readily ve

$$
(\hat{\phi}_{l+1} - \hat{\phi}_{l+2}, T_t^h + uT_x^h - S) = 0.
$$
\n(14)
\n
$$
(\hat{\phi}_{l+1} - \hat{\phi}_{l+2}, T_t^h + uT_x^h - S) = 0.
$$
\n(15)
\n
$$
(\hat{\phi}_{l+1} - \hat{\phi}_{l+2}, T_t^h + uT_x^h - S) = 0.
$$
\n(16)
\n
$$
(\hat{\phi}_{l+1} - \hat{\phi}_{l+2}, T_t^h + uT_x^h - S) = 0.
$$

 (x_l, L))
Definir $f(x)$
 $f(x) = f(x)$ $(t+1)$
= (t) Defining $\psi(x) = \int_{x_i}^x$ $\psi(x) = \phi_{l+2}(t) dt$, then it is readily verified that
 $\psi(x) = 6X^2(1-X), \quad X = (x - x_l)/h_N.$ ϕ $+1$

$$
\psi(x) = 6X^2(1-X), \quad X = (x - x_l)/h_N.
$$

Thus
$$
\psi(x_l) = \psi(L) = 0
$$
 and integrating (14) by parts leads to
\n
$$
\left(\psi, \frac{\partial}{\partial x}(T_l^h + u T_x^h - S)\right) = 0.
$$
\n(15)
\nFurthermore, since $\psi(x) > 0$ for $x \in (x_l, L)$, we may again apply the mean value theorem for integrals

 $(T_t^h$
*w*e do oth $(uT_x^h - S)$ = 0. (15)

may again apply the mean value theorem for integrals

s not identify the precise location of $\xi \in (x_l, L)$, but if

it may be shown that $\xi = L - 2h_N/5 + \mathcal{O}(h^2)$, so that Furthermore, since $\psi(x) > 0$ for $x \in (x_l, L)$, we may again apply the mean value theorem for integrals
to conclude that (12) holds. This argument does not identify the precise location of $\xi \in (x_l, L)$, but if
we assume tha to conclude that (12) holds. This argument does not identify the precise location of $\xi \in (x_1, L)$, but if \in (x_l , L), but if
 \in $\mathcal{O}(h^2)$, so that we assume that *S* and *u* are sufficiently smooth, it may be shown that $\xi = L - 2h_N/5 + \mathcal{O}(h^2)$, so that the former result is essentially recovered.
General case
We return now to the case of finite elements of general d the former result is essentially recovered.

General case

We return now to the case of finite elements of general degree $p \geq 1$ and establish the following result.

Theorem 1

If T^h is an FE function of the form (7) and *R* denotes the corresponding residual

$$
R(x,t) \equiv T_t^h + uT_x^h - \varepsilon T_{xx}^h - S, \quad x \in (x_l, L), \tag{16}
$$

then the 'no BC' FE equations (9) imply that for each $t > 0$ there is a point ξ

$$
(x, t) \equiv T_t^h + uT_x^h - \varepsilon T_{xx}^h - S, \quad x \in (x_l, L),
$$
\n
$$
(16)
$$
\nas (9) imply that for each $t > 0$ there is a point $\xi \in (x_l, L)$ where

\n
$$
\frac{\partial^{p-1}}{\partial x^{p-1}} (T_t^h + uT_x^h - S) \Big|_{x = \xi} = 0.
$$
\n(17)

\nnial in x of degree $\leq p$ and if u is a linear polynomial in x for $x \in (x_l, L)$.

 $^{-1}$ $(T_t^h$
degi Thermore, if *S* is a polynomial in *x* of degree $\leq p$

(d) for each $t > 0$, then

(i) the BC (17) holds at $\xi = L - [p/(2p + 1)]h$

(ii) the residual is zero at the *n* zeros $\frac{f}{f}$, of the $S = 0.$ (17)
and if *u* is a linear polynomial in *x* for $x \in (x_l, L)$ $\frac{1}{u}$ Furthermore, if *S* is a polynomial in *x* of degree $\leq p$ and if *u* is a linear polynomial in *x* for *x* and for each $t > 0$, then \in (x_l, L)
That is, and for each $t > 0$, then

-
- $L = L [p/(2p + 1)]h_N$
 Le p zeros { ξ_k } of the lions in the 'outflow κ
 $R(\xi_k, t) = 0$, (ii) the residual is zero at the *p* zeros { ξ_k } of the Radau polynomial of degree *p* on (x_l, L) . That is,
the finite element equations in the 'outflow element' imply
 $R(\xi_k, t) = 0, \quad k = 1, 2, ..., p,$ (18)
that they are equiv the finite element equations in the 'outflow element' imply

$$
R(\xi_k, t) = 0, \quad k = 1, 2, \dots, p,
$$
\n(18)

\nation at the Radau points in this element.

so that they are equivalent to collocation at the Radau points in this element.

Remarks

- 1. The assumption made in the second part of Theorem 1 that $S(x, t)$ be a polynomial in x of (x, t) be a polynomial in *x* of
common practice of projecting
r to facilitate the evaluation of
 $+1$)st and second derivatives
e can show that the 'boundary degree not exceeding p in (x_i, L) will be met if one adopts the common practice of projecting (x_l, L) will be met if one adopts the common practice of projecting
inderlying finite element basis in order to facilitate the evaluation of
tions on *S* and *u*, e.g. that their $(p + 1)$ st and second derivatives
o *x* be the source terms onto the underlying finite element basis in order to facilitate the evaluation of integrals.
- 2. With less restrictive conditions on *S* and *u*, e.g. that their $(p + 1)$ st and second derivatives $(p + 1)$ st and second derivatives
one can show that the 'boundary respectively with respect to *x* be continuous for $x \in (x_l, L)$, one can show that the 'boundary condition' (17) holds at $\xi = L - \frac{p}{2p+1} h_N + \mathcal{O}(h^2)$. condition' (17) holds at

$$
\xi = L - \frac{p}{2p+1} h_N + \mathcal{O}(h^2).
$$

3. The Radau polynomial $r_p(x)$ of degree *p* on (x_l, L) is defined by (see e.g. the book by Davis and Rabinowitz⁸)
 $r_p(x) = \frac{P_{p+1}(X) + P_p(X)}{1+X}$, $X = -1 + 2\frac{x - x_l}{h_N}$, $x \in (x_l, L)$, where $P_n(X)$ denotes the *n*th-degree L Rabinowitz⁸)

$$
r_p(x) = \frac{P_{p+1}(X) + P_p(X)}{1+X}, \quad X = -1 + 2\frac{x - x_l}{h_N}, \quad x \in (x_l, L),
$$

notes the *n*th-degree Legendre polynomial for $X \in (-1, 1)$.
to note that the point ξ at which (17) holds and the collocation
1 time (to $\mathcal{O}(h^2)$ in the general case).

where P_n

 \in (x_l, L) ,
-1, 1).
ollocation *X*) denotes the *n*th-degree Legendre polynomial for $X \in (-1, 1)$.
esting to note that the point ξ at which (17) holds and the collocat
for all time (to $\mathcal{O}(h^2)$ in the general case).
em 1 4. It is interesting to note that the point ξ at which (17) holds and the collocation points $\{\xi_k\}$ are stationary for all time (to $\mathcal{O}(h^2)$ in the general case).
 oof of Theorem 1

The proof depends on the pol stationary for all time (to $\mathcal{O}(h^2)$ in the general case).

f of Theorem 1

e proof depends on the polynomial
 $y_1(x) = (1 + X)^p(1 - X)$

Proof of Theorem 1

The proof depends on the polynomial

$$
\psi(x) = (1 + X)^p (1 - X)^{p-1},\tag{19}
$$

where, here and throughout the proof, *x* and *X* are related by

$$
\psi(x) = (1 + X)^p (1 - X)^{p-1},
$$
\nof, *x* and *X* are related by

\n
$$
X = -1 + 2 \frac{x - x_l}{h_N},
$$
\nif $(-1, 1)$.

\nif a polynomial of degree *p* that vanishes at $x = x_l(X = -1)$.

\n
$$
\{a_1, a_2, \ldots, a_p\} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{ such that}
$$
\n
$$
\frac{\partial^{p-1} u}{\partial p} = \frac{p}{h_N} \text{
$$

a linear map from $x \in (x_l, L)$ to $X \in (-1, 1)$.

 \in (*x_l*, *L*) to *X* \in (-1, 1).
 $\frac{\partial^{(p-1)} \psi}{\partial x^{(p-1)}}$ is a pol

are coefficients {*a*₁, *a*₂, .
 $\frac{\partial^{p-1} \psi}{\partial x^{p-1}}$ (*x* We first note that $\partial^{(p-1)}$
onsequently, there are co $\psi/\partial x^{(p-1)}$ is a polynomial of degree *p* that vanishes at $x = x_l(X = -1)$.
efficients $\{a_1, a_2, ..., a_p\}$ such that
 $\frac{\partial^{p-1}\psi}{\partial x^{p-1}}(x) = \sum_{j=1}^p a_j \phi_{j+l}(x)$,
 $-l$)th equation of (10) by a_i and summing, we find that Consequently, there are coefficients

$$
\frac{\partial^{p-1} \psi}{\partial x^{p-1}}(x) = \sum_{j=1}^p a_j \phi_{j+l}(x),
$$

ation of (10) by a_j and su

Consequently, there are coefficients
$$
\{a_1, a_2, ..., a_p\}
$$
 such that
\n
$$
\frac{\partial^{p-1} \psi}{\partial x^{p-1}}(x) = \sum_{j=1}^p a_j \phi_{j+l}(x),
$$
\nand by multiplying the $(j-l)$ th equation of (10) by a_j and summing, we find that\n
$$
\int_{x_l}^{L} \frac{\partial^{p+1} \psi}{\partial x^{p-1}}(T_l^h + u T_x^h - \varepsilon T_{xx}^h - S) dx = 0.
$$
\n(21)
\nWe next note that the polynomials\n
$$
\psi(x), \quad \frac{\partial^k \psi}{\partial x^k}(x), \quad k = 1, 2, ..., p-2,
$$

We next note that the polynomials

$$
\int_{x_l} \frac{\partial^k \psi}{\partial x^{p-1}} (T_l^h + u T_x^h - \varepsilon T_{xx}^h - S) dx = 0
$$

nials

$$
\psi(x), \quad \frac{\partial^k \psi}{\partial x^k}(x), \quad k = 1, 2, \dots, p-2
$$

all vanish at both *x*

$$
\psi(x), \quad \frac{\partial^k \psi}{\partial x^k}(x), \quad k = 1, 2, \dots, p - 2,
$$

\n
$$
x = L.
$$
 Thus, when we integrate (21) by $\int_{x_i}^{L} \psi \frac{\partial^{p-1}}{\partial x^{p-1}} (T_t^h + u T_x^h - \varepsilon T_{xx}^h - S) dx = 0.$
\n
$$
\text{ial of degree } p, \text{ its } (p + 1) \text{st derivative is}
$$

\n
$$
\int_{0}^{L} \psi \frac{\partial^{p-1}}{\partial x^{p-1}} (T_t^h + u T_x^h - S) dx = 0,
$$

 $x = x_l$ and $x = L$. Thus, when we integrate (21) by parts $p - 1$ times, we obtain
 $\int_{x_l}^{L} \psi \frac{\partial^{p-1}}{\partial x^{p-1}} (T_l^h + u T_x^h - \varepsilon T_{xx}^h - S) dx = 0$.

polynomial of degree *p*, its $(p + 1)$ st derivative is identically zero, so tha -1
gre Now, since *T^h* is a polynomial of degree *p*, its reduces to

of degree *p*, its
$$
(p + 1)
$$
st derivative is identically zero, so that this
\n
$$
\int_{x_l}^{L} \psi \frac{\partial^{p-1}}{\partial x^{p-1}} (T_l^h + u T_x^h - S) dx = 0,
$$
\n(22)
\n*h*, we invoke the mean value theorem for integrals to deduce that (17)

−1
ke
ply $(T_t^h$
the non- $+ uT_x^h - S)dx = 0,$ (22)

mean value theorem for integrals to deduce that (17)

iial of degree p, the second factor in the integrand of

t in the form and since $\psi(x) > 0$ for $x \in (x_l, L)$, we invoke the mean value theorem for integrals to deduce that (17) holds at some point $\xi \in (x_l, L)$.
When *u* is a linear function and *S* is a polynomial of degree *p*, the second fac holds at some point $\xi \in (x_1, L)$.

 \in (x_l, L) .
unction as
mial so y When *u* is a linear function and *S* is a polynomial of degree *p*, the second factor in the integrand of (22) is a linear polynomial so we may express it in the form

$$
\frac{\partial^{p-1}}{\partial x^{p-1}}(T_t^h + uT_x^h - S) = A(B - X).
$$

Substituting this into (22), using the definition of ψ and evaluating the resulting integrals gives

$$
B = \int_{-1}^{1} X \psi(X) dX \bigg/ \int_{-1}^{1} \psi(X) dX,
$$

$$
B = \frac{p+1}{2p+1}.
$$

so that

$$
B = \frac{p+1}{2p+1}
$$

Thus we have a zero at *X*

 $=\frac{p+1}{2p+1}$
 h (20), le
 s on the o

blynomial
 f freedom *B* which, through (20), leads to $\xi = L - ph_N/(2p + 1)$ as required.

the theorem depends on the observation that the residual *R* is a polynon

orthogonal to all polynomials of degree *p* which vanish at $x = x_l$ (beca

is one The proof of part (ii) of the theorem depends on the observation that the residual *R* is a polynomial of degree *p* that, by (10), is orthogonal to all polynomials of degree *p* which vanish at $x = x_i$ (because $= x_l$ (because
re that *R* were
gree p :⁸ of this last condition, there is one less degree of freedom than would be required to ensure that *R* were identically zero). Consequently, R must be a multiple of the Radau polynomial of degree p ⁸.

$$
R(x, t) = a(t)r_n(x), \quad x \in (x_l, L),
$$

for some function $a(t)$, and therefore (18) must hold.

x; $f(x, t) = a(t)r_p(x), \quad x \in (x_l, L),$
 x = (18) must hold.
 he 'no BC' finite element equal g of (1) on the interval $0 < x \leq 0$ (*t*), and therefore (18) must hold. \Box

heorem 1 that the 'no BC' finite element equations provide an approximation to

bblem consisting of (1) on the interval $0 < x \le \xi$, a Dirichlet condition at $x = 0$

pondition It follows from Theorem 1 that the 'no BC' finite element equations provide an approximation to the non-standard problem consisting of (1) on the interval $0 < x \leq \zeta$, a Dirichlet condition at $x = 0$ $= 0$
y be and the boundary condition

$$
\frac{\partial^{p-1}}{\partial x^{p-1}} (T_t + uT_x - S) = 0 \quad \text{at } x = \xi
$$

then *u* and *S* are sufficiently smooth,
condition at the outflow and this lea

 $\frac{-1}{u}$ $(T_t + uT_x - S) = 0$ at $x = \xi$
and *S* are sufficiently smooth,
on at the outflow and this lea
uT = $eT + S = 0 \le x \le \xi$ for some ξ
used to sin $\dot{\xi}(h, t) \in (x_l, L)$. When *u* and *S* are sufficiently smooth, the differential equation may be oplify the boundary condition at the outflow and this leads to the initial boundary value BVP)
 $T_t + uT_x = \varepsilon T_{xx} + S, \quad 0 < x \le \xi$ used to simplify the boundary condition at the outflow and this leads to the initial boundary value problem (IBVP)

Tt uTx ^e*Txx ^S*; ⁰ < *^x* 4 x; 1*T T*⁰; *^t* ⁰; *^t* > ⁰; @ *p*@*x p*

 $+1$
bo
s $(\xi, t) = 0, \quad t > 0.$

indary condition h:
 $(\varepsilon = 0)^{9,10}$ and also

irical experiments is (23)
ence
okes Discrete versions of this outflow boundary condition have been discussed in the finite difference $9,10$ and also in the context of the Navier–Stokes equations.^{2,3}

literature on hyperbolic problems ($\varepsilon = 0$)
equations.^{2,3}
We shall demonstrate through numerical e:
finite element method do indeed converge to
of (23) as describing the properties of equa
solutions of (23) lie from th We shall demonstrate through numerical experiments in Section 4 that the solutions of the 'no BC' finite element method do indeed converge to those of problem (23), so we may interpret the solutions of (23) as describing the properties of equations (9). We therefore need to quantify how far the solutions of (23) lie from those to the original problem posed on $0 < x < \infty$. We shall consider this issue in the next subsection.
3.1. Stationary problems
Our aim is to determine the difference between the solution to the issue in the next subsection.

3.1. Stationary problems

Our aim is to determine the difference between the solution to the problem

$$
\mathcal{S}^{\infty}\begin{cases}\nu T_x = \varepsilon T_{xx} + S, & 0 < x \leq \infty, \\
T(0) = 0, & T(x) \to \text{constant}, \quad x \to \infty,\n\end{cases}
$$
\n(24)

denoted by T^{∞} , and that of

$$
\mathcal{L}\left\{\n\begin{aligned}\nuT_x &= \varepsilon T_{xx} + S, \quad 0 < x \leq \xi, \\
T(0) &= 0, \\
\frac{\partial^k T}{\partial x^k}(\xi) &= 0, \\
\end{aligned}\n\right.\n\tag{25}
$$
\nconstant, then the general solution of $uT_x = \varepsilon T_{xx} + S$ satisfying

\n
$$
\frac{1}{2} \varepsilon^k = \frac{1}{2} \varepsilon^k
$$

 $\frac{\partial^2 \mathbf{r}}{\partial x^k}(\xi) = 0,$ stant, then t
 $(1) + \frac{1}{u} \int_0^x$ $T(0) = 0$ is given by

which we denote by
$$
T^{\xi}
$$
. If *u* is constant, then the general solution of $uT_x = \varepsilon T_{xx} + S$ satisfying
\n
$$
T(0) = 0
$$
 is given by
\n
$$
T(x) = A(e^{ux/\varepsilon} - 1) + \frac{1}{u} \int_0^x S(s) ds - \frac{1}{u} \int_0^x e^{u(x-s)/\varepsilon} S(s) ds
$$
\n(26)
\nfor any constant *A*.
\nIf $e^{-ux/\varepsilon} \int_0^x S(x) ds \to 0$ as $x \to \infty$, then $T(x) \to \text{constant}$ if the constant *A* is chosen such that
\n $A = A_{\infty}$, where

for any constant *A*.

(0) = 0 is given by
or any constant A.
If $e^{-ux/\varepsilon} \int_0^x S(x) dx$ If ey const
 $\frac{-ux/\varepsilon}{\delta} \int_0^x$
 ∞ , whe $\int_0^{\pi} S(x) ds \to 0$ as $x \to \infty$, then $T(x) \to \text{constant}$ if the constant *A* is chosen such that
ere
 $A_{\infty} = \int_0^{\infty} e^{-us/\varepsilon} S(s) ds.$
er with (26) defines T^{∞} . $A = A_{\infty}$, where

$$
A_{\infty} = \int_0^{\infty} e^{-us/\varepsilon} S(s) ds.
$$

This together with (26) defines T^{∞} .

 $A_{\infty} = \int_{0}^{\infty} e^{-us/\varepsilon} S(s) ds.$

is together with (26) defines T^{∞} .

The difference $e \equiv T^{\infty} - T^{\xi}$ satisfies the homogeneous difference $e \equiv T^{\infty} - T^{\xi}$ satisfies the homogeneous difference $e \equiv T^{\infty} - T^{\xi}$ at $\frac{1}{1}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ The difference $e \equiv T^{\infty} - T^{\zeta}$ satisfies the homogeneous differential equation $-ee^{\prime\prime} + ue^{\prime} = 0$ with

(b) = 0 and $d^{k}e/dx^{k} = d^{k}T^{\infty}/dx^{k}$ at $x = \xi$. Consequently,
 $e(x) = \left(\frac{\varepsilon}{u}\right)^{k} \frac{d_{k}T^{\infty}}{dx^{k}} (\xi)(e^{u(\xi$ *e*(0) = 0 and $d^k e/dx^k = d^k T^\infty/dx^k$ at $x = \xi$. Consequently,
 $e(x) = \left(\frac{\varepsilon}{u}\right)^k \frac{d_k T^\infty}{dx^k} (\xi) (e^{u(\xi - x)})$

for $0 \le x \le \xi$. It is readily shown that
 $d^k T^\infty = 1 \int_0^\infty e^{-(\xi - x)} d^k$

$$
e(x) = \left(\frac{\varepsilon}{u}\right)^k \frac{d_k T^{\infty}}{dx^k} (\xi) (e^{u(\xi - x)/\varepsilon} - e^{u\xi/\varepsilon})
$$

when that

for $0 \le x \le \xi$. It is readily shown that

$$
e(x) = \left(\frac{e}{u}\right)^{\alpha} \frac{a_k I}{dx^k} (\xi) (e^{u(\xi - x)/\varepsilon} - e^{u\xi/\varepsilon})
$$

\n
$$
\text{wn that}
$$

\n
$$
\frac{d^k T^{\infty}}{dx^k} (\xi) = \frac{1}{\xi} \int_{\xi}^{\infty} e^{u(\xi - s)/\varepsilon} \frac{d^{k-1} S}{ds^{k-1}} (s) ds.
$$

\n
$$
\text{wing lemma.}
$$

These results lead to the following lemma.

Lemma 1

solutions to problems \mathscr{S}^{∞} and \mathscr{S}^{ξ} differ by

If *u* is constant and *S* and its first *k* – 1 derivatives are continuous and bounded on
$$
(0, \infty)
$$
, then the
lutions to problems \mathcal{S}^{∞} and \mathcal{S}^{ξ} differ by

$$
T^{\infty}(x) - T^{\xi}(x) = \frac{1}{u} \left(\frac{\varepsilon}{u}\right)^{k-1} \int_{0}^{\infty} e^{-us/\varepsilon} \frac{d^{k-1}S}{ds^{k-1}} (s + \xi) ds (e^{-u(\xi - x)/\varepsilon} - e^{-u\xi/\varepsilon}) \qquad (27)
$$

$$
0 \le x \le \xi. \text{ Thus}
$$

$$
\max_{0 \le x \le \xi} |T^{\infty}(x) - T^{\xi}(x)| \le \frac{1}{u} \left(\frac{\varepsilon}{u}\right)^{k} \max_{\xi \le x \le \infty} \left| \frac{d^{k-1}S}{dt^{k-1}} (x) \right| \qquad (28)
$$

for $0 \leq x \leq \xi$. Thus

$$
\zeta(x) = \frac{1}{u} \left(\frac{\varepsilon}{u}\right)^{k-1} \int_0^\infty e^{-us/\varepsilon} \frac{d^{k-1}S}{ds^{k-1}} (s+\zeta) ds (e^{-u(\zeta-x)/\varepsilon} - e^{-u\zeta/\varepsilon})
$$
(27)

$$
\max_{0 \le x \le \zeta} |T^\infty(x) - T^\zeta(x)| \le \frac{1}{u} \left(\frac{\varepsilon}{u}\right)^k \max_{\zeta \le x \le \infty} \left|\frac{d^{k-1}S}{dx^{k-1}} (x)\right|
$$
(28)
left-hand side is achieved at $x = \zeta$.

and the maximum of the left-hand side is achieved at *x*

Remarks

neighbourhood of *x* = ξ .

he smoothness requirements on the source term *S* may be weakened (except in the neighbourhood of *x* = ξ) at the expense of introducing additional exponentially small terms into the right 1. The smoothness requirements on the source term *S* may be weakened (except in the the right sides of (27) and (28).

- 2. The requirement of *u* being constant may be relaxed (subject to $u(x) > 0$) at the expense of introducing more complex integrating factors. The results given are valid only if $u \neq 0$, so that some element of convection is essential in the problem.
- $(x) > 0$ at the expense of
valid only if $u \neq 0$, so that
vative boundary condition
er. If the no-flux condition
with linear $(k = p + 1 = 2)$ $\neq 0$, so that
y condition
x condition
 $p + 1 = 2$)
ces of $\mathcal{O}(e^2)$ 3. The most significant conclusion is that the higher the order of derivative boundary condition imposed at *x* = ξ , the closer the solutions *T*^{\oo} and *T*^{ξ} lie to each other. If the no-flux condition
d, we have $\xi = L$, $k = 1$ and a difference of $\mathcal{O}(\varepsilon)$, but with linear ($k = p + 1 = 2$)
($k = p + 1 = 3$) elements in the (2) is imposed, we have ξ
and quadratic $(k - n + 1)$ $L, k = 1$ and a difference of $\mathcal{O}(\varepsilon)$, but with linear $(k = p + 1 = 2)$
 $= 3$) elements in the 'no BC' formulation we get differences of $\mathcal{O}(\varepsilon^2)$
 \mathbf{e} first of these may be deduced as a special case of the r and quadratic $(k = p + 1 = 3)$ elements in the 'no BC' formulation we get differences of $\mathcal{O}(\varepsilon^2)$ $(k = p + 1 = 3)$ elements in the 'no BC' formulation we get differences of $\mathcal{O}(e^2)$ pectively (the first of these may be deduced as a special case of the results of reover, the last factor on the right of (27) represents and $\mathcal{O}(\varepsilon^3)$ respectively (the first of these may be deduced as a special case of the results of Lohéac¹). Moreover, the last factor on the right of (27) represents a term that is exponentially small outside a bo Lohéac¹). Moreover, the last factor on the right of (27) represents a term that is exponentially small outside a boundary layer of width $\mathcal{O}(\varepsilon)$ at $x = \xi$; any differences are therefore confined to this layer.
- 4. If the time-dependent problem (23) achieves a steady state as $t \to \infty$, then the results of
Lemma 1 also hold in that case for large times.
4. NUMERICAL RESULTS Lemma 1 also hold in that case for large times.

4. NUMERICAL RESULTS

We present three examples designed to confirm the results of the preceding sections.

Example 1. Steady problem

We choose $u = 1$ and consider the problem \mathscr{S}^{∞} with

$$
= 1
$$
 and consider the problem *9*[∞] with

$$
S(x) = \begin{cases} \sin(x), & 0 \le x \le \pi, \\ 0, & \pi < x < \infty. \end{cases}
$$

∞ are shown in Figure 1 for *ε* = 0·1 (broken line
problem on the truncated domain with *L* = 1 usi

The solutions *T*

 $S(x) = \begin{cases} \sin(x), & 0 \le x \le \pi, \\ 0, & \pi < x < \infty. \end{cases}$
 ∞ are shown in Figure 1 for $\varepsilon = 0.1$ (broken line

problem on the truncated domain with $L = 1$ usi

formulation (9) with linear and quadratic element

r by rescaling x, = 0.1 (broken line) and ε = 0.01 (full line).
ain with $L = 1$ using both the no-flux formuld
d quadratic elements. The choices of L and t
l cases we use ε = 1, 0.1, 0.01, 0.001 and a
ne computations were carried o We solve the problem on the truncated domain with $L = 1$ using both the no-flux formulation (8) = 1 using both the no-flux formulation (8)

: elements. The choices of *L* and *u* may be

use $\varepsilon = 1, 0.1, 0.01, 0.001$ and a uniform

tions were carried out in Matlab on a Sun

own in Figure 2: (a) $\varepsilon = 1$, (b) $\varepsilon =$ and the 'no BC' formulation (9) with linear and quadratic elements. The choices of *L* and *u* may be compensated for by rescaling x, ε and S. In all cases we use $\varepsilon = 1, 0.1, 0.01, 0.001$ and a uniform SPARCstation LX.

= 1, 0.1, 0.01, 0.001 and a uniform
were carried out in Matlab on a Sun
a Figure 2: (a) ε = 1, (b) ε = 0.1, (c)
the crosses denote the solution using
flux boundary condition (2). The 'no grid in space with $h = 2^{-j}$, *j*
SPARCstation LX.
The solutions for linear e
 $\varepsilon = 0.01$ and (d) $\varepsilon = 0.001$. T
'no BC' at $x = 1$ and the circ
BC' condition is seen to be = 2, 3, ..., 10. The computations were carried out in Matlab on a Sun
ements with $h = 1/32$ are shown in Figure 2: (a) $\varepsilon = 1$, (b) $\varepsilon = 0.1$, (c)
ee full line denotes the solution T^{∞} , the crosses denote the solut The solutions for linear elements with *h* $\varepsilon = 0.01$ and (d) $\varepsilon = 0.001$. The full line denotes the solution T^{∞} , the crosses denote the solution using ε = 0.01 and (d) ε = 0.001. The full line denotes the solution *T*[∞], the crosses denote the solution using 'no BC' at *x* = 1 and the circles denote the solution using the no-flux boundary condition (2). The 'no 'no BC' at *x* = 1 and the circles denote the solution using the no-flux boundary condition (2). The 'no
n is seen to be effective even at very large values of the diffusion coefficient and is
berior to the no-flux condition. BC' condition is seen to be effective even at very large values of the diffusion coefficient and is generally superior to the no-flux condition.

Figure 1. Solution T^{∞} to Example 1 with $\varepsilon = 0.1$ (broken line) and $\varepsilon = 0.01$ (full line). The vertical dotted line shows the location of the fictitious boundary location of the fictitious boundary

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Figure 2. T^{∞} (full line), T_{noflux}^h (circles) and T_{no}^h (crosses) for Example 1 with $h = 1/32$ and (a) $\varepsilon = 1$, (b) $\varepsilon = 0.1$, (c) $\varepsilon = 0.01$, (d) $\varepsilon = 0.001$ (c) $\varepsilon = 0.01$, (d) $\varepsilon = 0.001$
ex plot of the different

Each part of Figure 3 shows a log–log plot of the difference between a reference solution and a computed solution (evaluated at $x = 1 - h$) as a function of h for $\varepsilon = 1$ (crosses), $\varepsilon = 0.1$ (circles), $= 1 - h$) as a function of *h* for $\varepsilon = 1$ (crosses), $\varepsilon = 0.1$ (circles), (full line). In Figure 3(a) we show $T^{\infty} - T_{\text{noflux}}^h$, where T_{noflux}^h is t is clear that as $h \to 0$ the difference is proportional to ε e determined using equations (8). It is clear that as *h* = 0.01 (asterisks) and ε = 0.001 (full line). In Figure 3(a) we show $T^{\infty} - T_h^h$
etermined using equations (8). It is clear that as $h \to 0$ the difference is p
coordance with Lemma 1 with $k = 1$. Figure 3(b) shows t T_{noflux}^h , where T_{noflux}^h is \rightarrow 0 the difference is proportional to ε , in
nows the difference $T^{\infty} - T_{\text{nobc}}^h$ between the
nin and that produced by the linear 'no BC'
s as $\mathcal{O}(h^2)$ but asymptotes as $h \rightarrow 0$ to a value
a 1, this time with accordance with Lemma 1 with $k = 1$. Figure 3(b) shows the difference $T^{\infty} - T_{\text{nobe}}^h$ between the = 1. Figure 3(b) shows the difference $T^{\infty} - T_h^h$
he unbounded domain and that produced by the
e difference behaves as $\mathcal{O}(h^2)$ but asymptotes as *h*
ordance with Lemma 1, this time with $k = p + 1$
the difference bet exact solution to the problem on the unbounded domain and that produced by the linear 'no BC'

formulation. For small values of ε the difference behaves as $\mathcal{O}(h^2)$ but asymptotes as $h \to 0$ to a value
proportional to $\mathcal{O}(\varepsilon^2)$, again in accordance with Lemma 1, this time with $k = p + 1 = 2$.
Figures 3(c proportional to $\mathcal{O}(\varepsilon^2)$, again in accordance with Lemma 1, this time with $k = p + 1 = 2$.
Figures 3(c) and 3(d) compare the difference between T_{n}^h and T^{ξ} , where, from 1
 $\xi = 1 - h/3$. In Figure 3(c) the di Figures 3(c) and 3(d) compare the difference between T_{nobe}^h and T^{ξ} , where, from Theorem 1, $\xi = 1 - h/3$. In Figure 3(c) the difference tends to zero with h, but the rate of convergence, although close to two for larger values of h, ultimately diminishes to $\mathcal{O}(h)$. It is seen in Theorem 1 that the $= 1 - h/3$. In Figure 3(c) the difference tends to zero with *h*, but the rate of convergence, although ose to two for larger values of *h*, ultimately diminishes to $\mathcal{O}(h)$. It is seen in Theorem 1 that the ecise locat close to two for larger values of *h*, ultimately diminishes to $\mathcal{O}(h)$. It is seen in Theorem 1 that the precise location of ξ is only known when *S* has the same degree as the underlying FE space.
Accordingly, we precise location of ξ is only known when *S* has the same degree as the underlying FE space.
Accordingly we also compute the 'no BC' FEM with *S* replaced by its interpolant *S^h* defined by Accordingly, we also compute the 'no BC' FEM with *S* replaced by its interpolant *S^h* defined by

$$
S^h(x) = \sum_{j=0}^{N} S(x_j) \phi_j(x).
$$

shown in Figure 3(d)
nputations using quadr

 $=0$
n

The resulting difference $T^{\xi} - T^h_{\text{nobe}}$ is shown in Figure 3(d) and is seen to converge optimally at $\mathcal{O}(h^2)$.
The results of exactly analogous computations using quadratic finite elements are shown in Figure 4. T $O(h^2)$.
The
4. The
1 with The results of exactly analogous computations using quadratic finite elements are shown in Figure 4. They are again in agreement with the results of Theorem 1 (with $p = 2$, $\xi = 1 - 2h/5$) and Lemma 1 with $k = p + 1 = 3$. 1 with $k = p + 1 = 3$.

Figure 3. Example 1 with linear elements: (a) $T^{\infty} - T_{\text{noftux}}^h$, (b) $T^{\infty} - T_{\text{nobc}}^h$, (c) $T^{\xi} - T_{\text{nobc}}^h$ and (d) $T^{\xi} - T_{\text{nobc}}^h$ (with source term interpolated) as *h* varies. Key: crosses, $\varepsilon = 1$; circles $\frac{1}{3}$ $\overline{}$ s, interpolated) as *h* varies. Key: crosses, $\varepsilon = 1$; circles, $\varepsilon = 0.1$; asterisks, $\varepsilon = 0.01$; full line, $\varepsilon = 0.001$
direction of the state of the stat

Figure 4. As for Figure 3 but using quadratic elements

Example 2. Unsteady problem

This example is designed to test the assertion that the solutions of the 'no BC' equations (9) converge to those of the IBVP (23) as $h \to 0$. We choose initial data so as to give an exact solution of → 0. We choose initial data so as to give an exact solution of $S = 0$. The initial data are dependent on the degree *p*, the equently, on *h*. The exact solution is chosen as $\sum_{j=1,2}^{1/2\epsilon} (-1)^{j-1} \frac{\sin(\gamma_j x)}{\sin(\gamma_j)} e^{-(1$ equations (23) with *L* location $\xi = 1 - ph/(2p + 1)$ and, consequently, on h. The exact solution is chosen as

(23) with
$$
L = 1
$$
, $u = 1$ and $S = 0$. The initial data are dependent on the degree p , the
\n $= 1 - ph/(2p + 1)$ and, consequently, on h . The exact solution is chosen as
\n
$$
T(x, t) = \frac{2}{\varepsilon} e^{(x-1)/2\varepsilon} \sum_{j=1,2} (-1)^{j-1} \frac{\sin(\gamma_j x)}{\sin(\gamma_j)} e^{-(1-4\gamma_j^2 \varepsilon t)/4\varepsilon},
$$
\n(29)
\nsifies $T(0, t) = 0$, $T(1, 0) = 0$ and also the boundary condition $\partial^{p+1} T/\partial x^{p+1} = 0$ at $x = \xi$ if
\nare roots of
\n
$$
(1 - 4\varepsilon^2 \gamma^2) \sin(\gamma \xi) + 4\varepsilon \gamma \cos(\gamma \xi) = 0, \quad \xi = 1 - h/3, \quad p = 1,
$$
\n(30)
\n(1 - 12\varepsilon^2 \varepsilon^2) \sin(\gamma \xi) + 2\omega(2 - 4\varepsilon^2 \varepsilon^2) \cos(\gamma \xi) = 0, \quad \xi = 1 - h/3, \quad p = 1,\n(31)

 $=1,2$
also which satisfies $T(0, t) = 0$, $T(1, 0) = 0$ and also the boundary condition $\partial^{p+1}T/\partial x^{p+1} = 0$ at $x = \xi$ if γ_1 and γ_2 are roots of
 $(1 - 4\varepsilon^2 \gamma^2) \sin(\gamma \xi) + 4\varepsilon \gamma \cos(\gamma \xi) = 0$, $\xi = 1 - h/3$, $p = 1$, (30)
 $(1 - 12\vare$ γ_1 and γ_2 are roots of

$$
(1 - 4\varepsilon^2 \gamma^2) \sin(\gamma \zeta) + 4\varepsilon \gamma \cos(\gamma \zeta) = 0, \quad \zeta = 1 - h/3, \quad p = 1,
$$
 (30)

$$
(1 - 12\varepsilon^2 \gamma^2) \sin(\gamma \xi) + 2\varepsilon \gamma (3 - 4\varepsilon^2 \gamma^2) \cos(\gamma \xi) = 0, \quad \xi = 1 - 2h/5, \quad p = 2. \tag{31}
$$

 $(1 - 4\varepsilon^2 \gamma^2) \sin(\gamma \xi) + 4\varepsilon \gamma \cos(\gamma \xi) = 0, \quad \xi = 1 - h/3, \quad p = 1,$ (30)
 $\sin(\gamma \xi) + 2\varepsilon \gamma (3 - 4\varepsilon^2 \gamma^2) \cos(\gamma \xi) = 0, \quad \xi = 1 - 2h/5, \quad p = 2.$ (31)

smallest positive roots of these equations. The solutions are shown in Figure 5
 $(1 - 12\varepsilon^2 \gamma^2) \sin(\gamma \xi) + 2\varepsilon \gamma (3 - 4\varepsilon^2 \gamma^2) \cos(\gamma \xi) = 0, \quad \xi = 1 - 2h/5, \quad p = 2.$ (31)
ose the two smallest positive roots of these equations. The solutions are shown in Figure 5
 $h = 1/64$ and $\varepsilon = 0.1$ (left) and $\varepsilon = 0$ We choose the two smallest positive roots of these equations. The solutions are shown in Figure 5 for $p = 1$, $h = 1/64$ and $\varepsilon = 0.1$ (left) and $\varepsilon = 0.1$ (right). The presence of a boundary layer at $x = 1$ is evident for $\varepsilon = 0.01$. The solutions have a qualitatively similar form for $p = 2$.
The solution (29) defi is evident for $\varepsilon = 0.01$. The solutions have a qualitatively similar form for $p = 2$.

The solution (29) defines initial data

= 0.01. The solutions have a qualitatively similar form for
$$
p = 2
$$
.
\n(29) defines initial data
\n
$$
T_0(x) = \frac{2}{\varepsilon} e^{(x-1)/2\varepsilon} \sum_{j=1,2} (-1)^{j-1} \frac{\sin(\gamma_j x)}{\sin(\gamma_j)}
$$
\nand by extending this to be zero outside the interval, we may construct all line by\n
$$
T^{\infty}(x, t) = \frac{1}{\varepsilon} \int_0^1 T_0(s) e^{-(x-s-t)^2/4\varepsilon t} ds
$$

=1,2
:
side for $0 < x < 1$, and by extending this to be zero outside the interval, we may construct a solution of (1) on the entire real line by on the entire real line by

$$
e^{z} = j=1,2
$$
 $\sin(y_j)$
ing this to be zero outside the interval, we m

$$
T^{\infty}(x, t) = \frac{1}{\sqrt{(4\pi\epsilon t)}} \int_0^1 T_0(s) e^{-(x-s-t)^2/4\epsilon t} ds.
$$

when ε is small, T
solution that satisfies solution that satisfies $T(0, t) = 0$.

 $\int_0^\infty (x, t) = \frac{1}{\sqrt{(4\pi \varepsilon t)}} \int_0^1 T_0(s) e^{-(x-s-t)}$
exponentially small and so this exp
= 0.
that we present, the time integr
code based on variable order, variable of local express in time to a user s $\infty(0, t)$ is exponentially small and so this expression is sufficiently close to the es $T(0, t) = 0$.
al results that we present, the time integration of equations (9) has been
g a Matlab code based on variable order, var $(0, t) = 0.$
ssults that
Matlab coo
he level of
the level of
plutions us In the numerical results that we present, the time integration of equations (9) has been accomplished using a Matlab code based on variable order, variable step, backward differentiation formulae and controls the level of local errors in time to a user-specified tolerance. We have used a tolerance of 10^{-7} in our computations.

Figure 6 shows the solutions using linear elements at $t = \varepsilon$ with $h = 1/64$ and (a) $\varepsilon = 0.1$ and (b) = ε with $h = 1/64$ and (a) $\varepsilon = 0.1$ and (b)
is shown). The circles denote the solution
en by (29) and the dotted line denotes T^{∞} .
1 both cases, whereas there is a notable $\varepsilon = 0.01$ (only the solution in the interval $0.75 \le x \le 1$ is shown). The circles denote the solution T_{noth}^h , the crosses denote T_{nobb}^h , the full line denotes T given by (29) and the dotted line denotes T T_{noflux}^h , the crosses denote T_{nobo}^h , the full line denotes *T* given by (29) and the dotted line denotes T_{∞}^{∞} .
There is very close agreement between T_{nobc}^h and *T* in both cases, whereas there is There is very close agreement between T_{nobe}^h and *T* in both cases, whereas there is a notable

Figure 5. Example 2: Exact solutions of IBVP (23) with $p = 1$, $h = 1/64$ and $\varepsilon = 0.1$ (left) and $\varepsilon = 0.01$ (right) $p = 1$, $h = 1/64$ and $\varepsilon = 0.1$ (left) and $\varepsilon = 0.01$ (right)

The corresponding results for quadratic elements are shown in Figure 7. The conclusions to be drawn are the same as for linear elements. Figure 6. Example 2: solutions using linear elements at $t = \varepsilon$ with $h + 1/64$ and (a) $\varepsilon = 0.1$ and (b) $\varepsilon = 0.01$ (only solution in interval $0.75 \le x \le 1$ is shown). Key; circles, T_{noflux}^h ; crosses, T_{nobc}^h ; interval $0.75 \le x \le 1$ is shown). Key; circles, T_{noflux}^h ; crosses, T_{nobc}^h ; full line, *T*; dotted line, *T*
esponding results for quadratic elements are shown in Figure 7. The conci-
the same as for linear eleme $-75 \le x \le 1$ is shown). Key; circles, T_h^h
ng results for quadratic elemente as for linear elements. ...
lu

Quantitative measures of the differences $T^{\infty} - T_{\text{noltux}}^h$, $T^{\infty} - T_{\text{nobe}}^h$ and $T - T_{\text{nobe}}^h$ as functions of *h* p presented in Figure 8 (Linear) and Figure 9 (quadratic). Results are given for $\varepsilon = 0.1$ (circle $\frac{9}{T_{\text{as}}}$ R
R
Fi are presented in Figure 8 (Linear) and Figure 9 (quadratic). Results are given for $\varepsilon = 0.1$ (circles) and $\varepsilon = 0.01$ (crosses). It is seen that both $T^{\infty} - T_{\text{noflux}}^h$ (Figures 8(a) and 9(a)) and $T^{\infty} - T_{\text{nobc}}^h$ (۔
و = 0.01 (crosses). It is seen that both $T^{\infty} - T_{\text{noflux}}^h$ (Figures 8(a) and 9(a)) and $T^{\infty} - T_{\text{nobc}}^h$ (Figures

(b) and 9(b) behave as $\mathcal{O}(\varepsilon)$ as $h \to 0$ whereas $T - T_{\text{nobc}}^h$ (Figure 8(c) and 9(c)) behaves as $\frac{1}{2}$
 $\frac{1}{2}$ 8(b) and 9(b) behave as $\mathcal{O}(e)$ as $h \to 0$ whereas $T - T_{\text{nobe}}^h$ (Figure 8(c) and 9(c)) behaves as $\mathcal{O}(h^2)$ for both linear and quadratic elements, suggesting that the solutions of the 'no BC' FE equations do inde both linear and quadratic elements, suggesting that the solutions of the 'no BC' FE equations do indeed converge to those of (23).

This example does not properly reflect the ability of the various FE methods to approximate T^{∞} $\frac{1}{2}$ because of the boundary layer in the neighbourhood of $x = 1$. We therefore include one further example which does not contain any such layers.
Example 3. Convection of a Gaussian
We require a smooth exact solution T^{\in example which does not contain any such layers.

Example 3. Convection of a Gaussian

We require a smooth exact solution
$$
T^{\infty}
$$
 on $0 < x < \infty$ and this is to
\n
$$
T^{\infty}(x, t) = \frac{1}{\sqrt{(4\pi\varepsilon(t + t_0))}} e^{-(x - x_0 - t)^2/4\varepsilon(t + t_0)}
$$

 ∞ and this is taken to be
 $\int_{0}^{-(x-x_0-t)^2/4\varepsilon(t+t_0)}$

e take $x_0 = 0.5$ and integrativer $0 < t < 0.4$. These has $^{\infty}(x, t) = \frac{1}{\sqrt{(4\pi\varepsilon(t + t_0))}} e^{-(x - x_0 - t)^2/4\varepsilon(t + t_0)}$

of ε and x_0 . For $\varepsilon = 0.1$ we take $x_0 = 0.5$
 $x_0 = 0.75$ and integrate over $0 < t < 0.4$

olution are convected out of the domain is

gure 10 with $\varepsilon = 0.$ with $t_0 = 0.1$ and two choices of ε and x_0 . For $\varepsilon = 0.1$ we take $x_0 = 0.5$ and integrate over $0 < t < 1$, whereas for $\varepsilon = 0.01$ we take $x_0 = 0.75$ and integrate over $0 < t < 0.4$. These have been chosen so that whereas for $\varepsilon = 0.01$ we take $x_0 = 0.75$ and integrate over $0 < t < 0.4$. These have been chosen so that the major features of the solution are convected out of the domain in the specified time intervals.
The solutions a that the major features of the solution are convected out of the domain in the specified time intervals.

Figure 7. As for Figure 6 but using quadratic elements

Figure 8. Example 2 with linear elements: (a) T^{∞} $\frac{1}{2}$

Figure 9. As for Figure 8 but using quadratic elements

The grids have used $h = 1/32$ for linear elements and $h = 1/16$ for quadratic elements (so that both systems have the same number of degrees of freedom). Reducing the size of *h* in both cases leads to virtually identical results (to graphical accuracy), since the dominant factor in the error is the value of ε .

 $\epsilon = 0.1$ and Figure 11(b) for $\epsilon = 0.01$. Th circles, crosses and full line denote the solutions The solutions at the outflow boundary $x = 1$ for linear elements are shown in Figure 11(a) for = 1 for linear elements are shown in Figure 11(a) for
h circles, crosses and full line denote the solutions
ralues of ε the 'no BC' formulation is clearly superior to
lements are shown in Figure 12. The magnitude of t $\varepsilon = 0.1$ and Figure 11(b) for $\varepsilon = 0.01$. Th circles, crosses and full line denote the solutions T_{noflux}^h , T_{nobc}^h and T^{∞} respectively. For both values of ε the 'no BC' formulation is clearly superio T_{noflux}^h , T_{nobc}^h and *T*

the no-flux bound: the no-flux boundary conditions.

 T_{noflux}^h , T_{nobc}^h and T^{∞} respectively. For both values of ε the 'no BC' formulation is clearly superior to the no-flux boundary conditions.
The corresponding results for quadratic elements are shown in The corresponding results for quadratic elements are shown in Figure 12. The magnitude of the shows a marginal improvement over the use of linear elements.

Figure 10. Example 3: exact solutions with $\varepsilon = 0.1$ (left) and $\varepsilon = 0.01$ (right)
 $\varepsilon = 0.01$ (right)

Figure 11. Example 3: solutions using linear element at $x = 1$ with $h = 1/32$ and (a) $\varepsilon = 0.1$ and (b) $\varepsilon = 0.01$. Key: circles,
 T_{noflux}^h ; crosses, T_{nobc}^h ; full line, T^{∞}
5. AN EQUIVALENT DIRICHLET-NEUMA T_{noflux}^h ; crosses, T_{nobc}^h ; full line, *T*

5. AN EQUIVALENT DIRICHLET–NEUMANN PROBLEM $\frac{1}{\sqrt{1}}$

It was shown in Section 3 that the 'no BC' finite element equations (9) approximate the IBVP (23) in which there features a boundary condition involving the $(p + 1)$ st derivative of T. This leads to a non- $(p + 1)$ st derivative of *T*. This leads to a non-
n-singular. We shall show that this difficulty
D' finite element equations to the standard
D. The process will be sketched for FE spaces
 $= 1$ and 2. standard problem for which no theory has been developed regarding either convergence or, in steady cases, whether the linear equations they generate are non-singular. We shall show that this difficulty may be partially circumvented by relating the 'no BC' finite element equations to the standard Galerkin FE approximation of a more standard problem. The process will be sketched for FE spaces of arbitrary degree *p* and details given for the cases $p = 1$ and 2.

 $=$ 1 and 2.
enable us t
aations may
denote a po We begin with some preliminary results that will enable us to determine a combination of test functions to replace ϕ_{pN} in (9) so that the 'no BC' equations may be reorganized to resemble those
for a Dirichlet-Neumann problem. For each n let Ψ denote a polynomial of degree 2n (which we for a Dirichlet–Neumann problem. For each p let Ψ denote a polynomial of degree $2p$ (which we shall specify below) and define

$$
\psi(x) = \frac{\mathrm{d}^p \Psi}{\mathrm{d} x^p}(x). \tag{32}
$$

 $\psi(x) = \frac{d^p \Psi}{dx^p}(x).$ (32)

a arbitrary finite element function, it may be shown by
 $\psi^{h} = \sum_{k=0}^{p} (-1)^j \frac{\partial^j V^h}{\partial x^j} \left(\varepsilon \frac{\partial^{p-j+1} \Psi}{\partial x^j} + u \frac{\partial^{p-j} \Psi}{\partial x^j} \right)^k$ Then, if $V^h = \sum$
integrating by par
 $\int_{x_t}^{L} \psi(-\varepsilon V_{xx}^h)$ integrating by parts that

if
$$
V^h = \sum_1 p^N V^h_j \phi_j(x)
$$
 denotes an arbitrary finite element function, it may be shown by
ating by parts that

$$
\int_{x_i}^L \psi(-\varepsilon V^h_{xx} + uV^h_x) dx = u \frac{\partial^{p-1} \Psi}{\partial x^{p-1}} V^h_x \Big|_{x=x_i}^L - \sum_{j=2}^p (-1)^j \frac{\partial^j V^h}{\partial x^j} \left(\varepsilon \frac{\partial^{p-j+1} \Psi}{\partial x^{p-j+1}} + u \frac{\partial^{p-j} \Psi}{\partial x^{p-j}} \right) \Big|_{x=x_i}^L.
$$
(a)
1.5

Figure 12. As for Figure 11 but using quadratic elements

We now impose the
$$
2p + 1
$$
 interpolation conditions
\n
$$
\frac{d^j \Psi}{dx^j}(x_l) = 0, \quad j = 0, 1, ..., p,
$$
\n
$$
\frac{d^{p-j} \Psi}{dx^{p-j}}(L) = (-1)^{j+1} \left(\frac{\varepsilon}{u}\right)^j, \quad j = 1, 2, ..., p,
$$
\nwhich uniquely define $\Psi(x)$, and we obtain

which uniquely define $\Psi(x)$, and we obtain

$$
\frac{1}{j}(L) = (-1)^{j+1} \left(\frac{v}{u}\right), \quad j = 1, 2, ..., p,
$$
\nwe obtain\n
$$
\int_{x_i}^{L} \psi(-\varepsilon V_{xx}^h + uV_x^h) dx = \varepsilon V_x^h(L).
$$
\n(i33)\n\nin a function of degree p on (x_i, L) with $\psi(x_i) = 0$, there are coefficients $\{a_j\}$

which uniquely define $\Psi(x)$, and we obtain
 $\int_{x_i}^{L} \psi(-\varepsilon V) \, dV$
Moreover, since $\psi(x)$ is a polynomial of deg Moreover, since $\psi(x)$ is a polynomial of degree *p* on (x_l, L) with $\psi(x_l) = 0$, there are coefficients $\{a_j\}$
such that
 $\psi(x) = \sum_{j=1}^p a_j \phi_{j+l}(x)$, such that

$$
\psi(x) = \sum_{j=1}^{p} a_j \phi_{j+l}(x),
$$

ctions $\{\phi_j\}$ whose sup
n the steady case and
adv case Multiplying

 $\begin{align} \n\phi_j \n\end{align}$ $\frac{1}{2}$
SE

a linear combination of the FE basis functions $\{\phi_j\}$ whose support is restricted to the outflow element (x_l, L) .
The essential features are contained in the steady case and so we set all time derivatives to zero—
we wi (x_l, L) .
The we will over *j* The essential features are contained in the steady case and so we set all time derivatives to zero we will subsequently return to the unsteady case. Multiplying the *j*th equation of (10) by *aj*, summing

over *j* and adding the result to the last equation of (10), we obtain, since
$$
p + l = pN
$$
,
\n
$$
\int_{x_l}^{L} (\phi_{pN} + \psi)(uT_x^h - \varepsilon T_{xx}^h - S)dx = 0.
$$
\nReorganizing this and using (33) with $V^h = T^h$, we find

Reorganizing this and using (33) with *V^h*

$$
\int_{x_l} (\phi_{pN} + \psi)(uT_x^h - \varepsilon T_{xx}^h - S)dx = 0.
$$

ng (33) with $V^h = T^h$, we find

$$
\int_{s_l}^L \phi_{pN}(uT_x^h - \varepsilon T_{xx}^h - S)dx + \varepsilon T_x^h(L) = \int_{x_l}^L \psi S ds.
$$

ond-derivative term by parts and use $\phi_{pN}(L) = 1$

$$
\int_{x_l}^L (\phi_{pN}uT_x^h + \varepsilon \phi_{pN_x}T_x^h - \phi_{pN}S)dx = \int_{x_l}^L \psi S ds.
$$

We now integrate the second-derivative term by parts and use ϕ_{pN}

and-derivative term by parts and use
$$
\phi_{pN}(L) = 1
$$
 to give

\n
$$
\int_{x_l}^{L} (\phi_{pN} u T_x^h + \varepsilon \phi_{pN_x} T_x^h - \phi_{pN} S) dx = \int_{x_l}^{L} \psi S ds.
$$
\n(34)

\n
$$
\dots, pN - 1, \text{ we have now shown that the steady 'no BC' finite element}
$$
\nbut to

\n
$$
u T_x^h) + \varepsilon(\phi_i, T_x^h) = (\phi_i, S), \quad j = 1, 2, \dots, pN - 1,
$$

Since $\phi_j(L) = 0, j = 1, 2, ..., pN - 1$, we have now shown that the steady 'no BC' finite element equations (9) are equivalent to
 $(\phi_j, uT_x^h) + \varepsilon(\phi_{j_x}, T_x^h) = (\phi_j, S), \quad j = 1, 2, ..., pN - 1,$
 $(\phi_j, uT_x^h) + \varepsilon(\phi_{j_x}, T_x^h) = (\phi_j, S) + \int_{s}^{L} \$ equations (9) are equivalent to

$$
(\phi_j, u_i^n) + \varepsilon(\phi_{j_x}, T_x^n) = (\phi_j, S), \quad j = 1, 2, \dots, pN - 1,
$$

\n
$$
(\phi_j, u_i^n) + \varepsilon(\phi_{j_x}, T_x^n) = (\phi_j, S) + \int_{x_i}^L \psi S ds, \quad j = pN.
$$

\n
$$
\text{Faleckin FE approximation of the Dirichlet-Neumann pr}
$$

\n
$$
-\varepsilon T_{xx} + u_i^n = S, \quad 0 < x < L,
$$

\n
$$
T(0) = 0.
$$

This is the standard Galerkin FE approximation of the Dirichlet–Neumann problem

$$
-\varepsilon T_{xx} + uT_x = S, \quad 0 < x < L, T(0) = 0, uT_x(L) = \alpha,
$$
\n
$$
(35)
$$

where

$$
\alpha = \frac{u}{\varepsilon} \int_{x_l}^{L} \psi S \, \mathrm{d}s
$$
\n
$$
\text{that } \psi = 2\varepsilon(x - \psi) - (h_N/3) S_x(L)
$$

clearly depends on *h*.

For linear elements it is readily shown that $\psi = 2\varepsilon(x - x_l)/uh_N^2$ and, by Taylor expansions,
 $\alpha = S(L) - (h_N/3)S_x(L) + \mathcal{O}(h^2)$
 $= S(L - h_N/3) + \mathcal{O}(h^2)$,

aich is independent of ε . For quadratic elements we find

$$
\alpha = S(L) - (h_N/3)S_x(L) + \mathcal{O}(h^2)
$$

= $S(L - h_N/3) + \mathcal{O}(h^2)$,
adratic elements we find

$$
\frac{1}{2} \left[\frac{\varepsilon}{nh_m} \left(2 \frac{x - x_l}{h_m} - 1 \right) + \frac{\varepsilon^2}{n^2 h^2} \right]
$$

which is independent of ε . For quadratic elements we find

$$
= S(L - h_N/3) + \mathcal{O}(h^2),
$$

nt of ε . For quadratic elements we find

$$
\psi(x) = 6 \frac{x - x_l}{h_N} \left[\frac{\varepsilon}{uh_N} \left(2 \frac{x - x_l}{h_N} - 1 \right) + \frac{\varepsilon^2}{u^2 h_N^2} \left(3 \frac{x - x_l}{h_N} - 4 \right) \right]
$$

$$
\alpha = S(L) + \frac{\varepsilon}{u} S_x(L) + \mathcal{O}(h^2 + \varepsilon h),
$$

and

$$
\lfloor uh_N \setminus h_N \rfloor - u^2 h_N^2 \setminus
$$

$$
\alpha = S(L) + \frac{\varepsilon}{u} S_x(L) + \mathcal{O}(h^2 + \varepsilon h),
$$

whereas for finite elements of degree *p* it may be shown that

$$
\alpha = S(L) + \frac{c}{u} S_x(L) + \mathcal{O}(h^2 + \varepsilon h),
$$

or finite elements of degree *p* it may be shown that

$$
\alpha = S(L) + \frac{\varepsilon}{u} S_x(L) + \dots + \left(\frac{\varepsilon}{u}\right)^{p-1} \frac{\partial^{p-1} S}{\partial x^{p-1}}(L) + \mathcal{O}(h^p + \varepsilon h^{p-1} + \dots + \varepsilon^{p-1} h).
$$
(36)
relate the implied boundary condition $uT_x = \alpha$ to the solution of the original problem on
finite real line, we note that the exact solution T^{∞} satisfies the differential equation

$$
T_x^{\infty} - \frac{\varepsilon}{u} T_{xx}^{\infty} = \frac{1}{u} S
$$

 $S(L) + \frac{S}{u}S_x(L) + \cdots + \left(\frac{S}{u}\right)$
late the implied boundary content that the real line, we note that the \overline{u}
so. In order to relate the implied boundary condition $uT_x = \alpha$ to the solution of the original problem on
the semi-infinite real line, we note that the exact solution T^{∞} satisfies the differential equation
 $T^{\infty}_x - \frac{\v$ the semi-infinite real line, we note that the exact solution T^{∞} satisfies the differential equation
 $T_x^{\infty} - \frac{\varepsilon}{u} T_x^{\infty} = \frac{1}{u} S$

$$
T_x^{\infty} - \frac{\varepsilon}{u} T_{xx}^{\infty} = \frac{1}{u} S
$$

as by the different

$$
\frac{\partial}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \right)_{x=1}^{u}
$$

$$
T_x^{\infty} - \frac{\varepsilon}{u} T_x^{\infty} = \frac{1}{u} S
$$

on $0 < x < \infty$ and multiplying both sides by the differential operator

$$
1 + \frac{\varepsilon}{u} \frac{\partial}{\partial x} + \dots + \left(\frac{\varepsilon}{u}\right)^{p-1} \frac{\partial^{p-1}}{\partial x^{p-1}},
$$

we find that

we find that

$$
u \frac{\partial}{\partial x} \left\{ u \right\} = \frac{1}{2} \left(\frac{1}{u} \right)^{p-1} \frac{\partial^{p-1} S}{\partial x^{p-1}}(L) + u \left(\frac{\varepsilon}{u} \right)^{p} \frac{\partial^{p+1} T^{\infty}}{\partial x^{p+1}}(L)
$$
\n
$$
\text{ufficiently smooth. Thus, in the limit } h \to 0,
$$
\n
$$
u T_x^{\infty}(L) = \alpha + u \left(\frac{\varepsilon}{u} \right)^{p} \frac{\partial^{p+1} T^{\infty}}{\partial x^{p+1}}(L),
$$
\n
$$
\text{where } \alpha \in \mathbb{R} \text{ and } \alpha \in \mathbb{R} \text
$$

provided that *S* is sufficiently smooth. Thus, in the limit *h*

$$
S(L) + \cdots + \left(\frac{a}{u}\right)^{p-1} \frac{\partial}{\partial x^{p-1}}(L) + u\left(\frac{a}{u}\right)
$$

moon both. Thus, in the limit $h \to 0$,

$$
uT_x^{\infty}(L) = \alpha + u\left(\frac{\varepsilon}{u}\right)^p \frac{\partial^{p+1} T^{\infty}}{\partial x^{p+1}}(L)
$$

uivalent, in this limit, to the BC ∂

 $\frac{1}{x}$
le $(L) = \alpha + u \left(\frac{u}{u} \right)$
 *u*t, in this limi
 $u = T^{\infty} - T$, v
 $e_{xx} + ue_x = 0$. $\frac{1}{\partial x^{p+1}}(L)$,
b the BC ∂^L
re T is a so:
 $0 \le x \le L$ $+1$
e I
'is so that the BC $uT_x(L) = \alpha$ is equivalent, in this limit, to the BC $\partial^{p+1}T/\partial x^{p+1} = 0$ used in (23) (see
also (4) with $j = p + 1$).
It now follows that the difference $e = T^{\infty} - T$, where T is a solution of (35), satisfies
 also (4) with $j = p + 1$). $=p+1$).
ws that the

It now follows that the difference
$$
e = T^{\infty} - T
$$
, where *T* is a solution of (35), satisfies
\n
$$
-\varepsilon e_{xx} + u e_x = 0, \quad 0 < x < L
$$
\n
$$
e(0) = 0,
$$
\n
$$
u e_x(L) = \delta,
$$
\nhere $\delta = \mathcal{O}((\varepsilon + h)^p)$. By the same reasoning that led to Lemma 1, we may conclude that\n
$$
T^{\infty} - T = \mathcal{O}(\varepsilon(\varepsilon + h)^p).
$$

where $\delta = \mathcal{O}((\varepsilon + h)^p)$. By the same reasoning that led to Lemma 1, we may conclude that
 $T^{\infty} - T = \mathcal{O}(\varepsilon(\varepsilon + h)^p)$.

$$
T^{\infty} - T = \mathcal{O}(\varepsilon(\varepsilon + h)^p).
$$

The standard Galerkin method will, subject to smoothness of the data, generate a numerical solution

$$
T^{\infty} - T^h = \mathcal{O}(\varepsilon(\varepsilon + h)^p + h^{p+1}).
$$
\n(38)

Th that is within $\mathcal{O}(h^{p+1})$ of *T*. Consequently,
 $T^{\infty} - T^h =$
The error is therefore $\mathcal{O}(h^{p+1})$ when $\varepsilon \ll h$ and
found in Example 1 (see Figures 3 and 4).
The same construction annlies to unsteady $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $-T^h = \mathcal{O}(\varepsilon(\varepsilon + h)^p + h^{p+1}).$ (38)
 $\leq k/h$ and $\mathcal{O}(\varepsilon^{p+1})$ when $h \ll \varepsilon$, which is exactly the behaviour

and 4).

Insteady problems and may be shown to lead to the IBVP
 $\mu Tx = \varepsilon T + S$ ($0 \leq x \leq L$) The error is therefore $\mathcal{O}(h^{p+1})$ when $e \ll h$ and $\mathcal{O}(e^{p+1})$ when $h \ll e$, which is exactly the behaviour found in Example 1 (see Figures 3 and 4).
The same construction applies to unsteady problems and may be show found in Example 1 (see Figures 3 and 4).

The same construction applies to unsteady problems and may be shown to lead to the IBVP

$$
T_t + uTx = \varepsilon T_{xx} + S, \quad 0 < x < L,
$$
\n
$$
T(0, t) = 0,
$$
\n
$$
uT_x(L, t) = \alpha,
$$
\n
$$
\alpha = \frac{u}{\varepsilon} \int_0^L \psi(s - T) \, ds
$$

where

$$
(L, t) = \alpha,
$$

$$
\alpha = \frac{u}{\varepsilon} \int_{x_l}^{L} \psi(S - T_t) \mathrm{d}s
$$

 $=\frac{u}{\varepsilon}$

uce nto $\psi(S - T_t)$ d*s*
amount of c
inite element
 $\psi(t) = \alpha$ may and ψ retains its earlier definition. To reduce the amount of detail, we assume that *S* is a polynomial of degree *p* in *x* (or that it is interpolated into the finite element basis) and discuss the cases $p = 1$ and
2.

For $p = 1$ it may be shown that the BC $uT_x(L, t) = \alpha$ may be written as
 $T_t + uT_x = S - \frac{1}{3} h_N(S_x - T_{xt})$

at 2.

For $p = 1$ it may be shown that the BC $uT_r(L, t) = \alpha$ may be written as

$$
T_t + uT_x = S - \frac{1}{3} h_N(S_x - T_{xt})
$$

= 1 it may be shown that the BC $uT_x(L, t) = \alpha$ may be written as
 $T_t + uT_x = S - \frac{1}{3} h_N(S_x - T_{xt})$

, so that as $h \to 0$ we obtain the condition $T_t + uT_x = S$ (cf (3))
 $\begin{aligned} x_x(L, t) &= 0. \end{aligned}$
 $\begin{aligned} -2$ the corresponding BC is *uT_x* = $S - \frac{1}{3} h_N (S_x - T_{xt})$
 n the condition $T_t + uT_x =$ at $x = L$, so that as $h \to 0$ we obtain the condition $T_t + uT_x = S$ (cf (3)), which is equivalent to
setting $T_{xx}(L, t) = 0$.
For $p = 2$ the corresponding BC is
 $T_t + uT_x = S - \frac{\varepsilon}{u}(S_x - T_{xt}) - \frac{1}{20}h_N^2(S_{xx} - T_{xt}).$ (39) setting $T_{xx}(L, t) = 0$. $(L, t) = 0.$
2 the corn
mutiating (1)

For $p = 2$ the corresponding BC is = 2 the corresponding BC is
 $T_t + uT_x$ =
rentiating (1) and eliminating

$$
T_t + uT_x = S - \frac{\varepsilon}{u} (S_x - T_{xt}) - \frac{1}{20} h_N^2 (S_{xx} - T_{xxt}).
$$
\n(39)
\neliminating T_{xx} , it may be shown that T^{∞} satisfies
\n
$$
T_t^{\infty} + uT_x^{\infty} = S - \frac{\varepsilon}{u} (S_x - T_{xt}^{\infty}) - u \left(\frac{\varepsilon}{u}\right)^2 T_{xxx}^{\infty}
$$

$$
T_t + uT_x = S - \frac{c}{u}(S_x - T_{xt}) - \frac{1}{20}h_N^2(S_{xx} - T_{xxt}).
$$

By differentiating (1) and eliminating T_{xx} , it may be shown that T^{∞} satisfies

$$
T_t^{\infty} + uT_x^{\infty} = S - \frac{\varepsilon}{u}(S_x - T_x^{\infty}) - u(\frac{\varepsilon}{u})^2 T_{xxx}^{\infty}
$$

 \mapsto $\frac{1}{1}$ and we see that in the limit *h* $T_{xxx}(L, t) = 0.$

 $uT_x^{\infty} = S - \frac{S}{u}(S_x - T_x^{\infty}) - u(\frac{S}{u})$
0 the implied BC (39) for quadrated
dopted for *p* > 2 to show that the
element approximation of (1) on → 0 the implied BC (39) for quadratic elements is equivalent to setting

e adopted for $p > 2$ to show that the 'no BC' finite element equations

te element approximation of (1) on $0 < x < L$ with the 'extrapolation'
 x^{p+1 $(L, t) = 0.$
 L similar a verge to the ndary converge to the neutron of the neutron of the report A similar approach may be adopted for $p > 2$ to show that the 'no BC' finite element equations
nyerge to the Galerkin finite element approximation of (1) on $0 < x < I$ with the 'extrapolation' converge to the Galerkin finite element approximation of (1) on $0 < x < L$ with the 'extrapolation' boundary condition $\partial^{p+1}T/\partial x^{p+1} = 0$ at $x = L$.

 $x^{1}T/\partial x^{p+1} = 0$ at $x = L$.
arguments used in this se
a future paper. We believe that the arguments used in this section may be extended to higher dimensions and this will be reported on in a future paper.

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